# A (ROUGH) PATHWISE APPROACH TO A CLASS OF NONLINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We consider nonlinear parabolic evolution equations of the form $\partial_t u = F\left(t, x, Du, D^2u\right)$, subject to noise of the form $H\left(x, Du\right) \circ dB$ where $H$ is linear in $Du$ and $\circ dB$ denotes the Stratonovich differential of a multidimensional Brownian motion. Motivated by the essentially pathwise results of [Lions, P.-L. and Souganidis, P.E.; Fully nonlinear stochastic partial differential equations. C. R. Acad. Sci. Paris Sér. I Math. 326 (1998), no. 9] we propose the use of rough path analysis [Lyons, T. J.; Differential equations driven by rough signals. Rev. Mat. Iberoamericana 14 (1998), no. 2, 215–310] in this context. Although the core arguments are entirely deterministic, a continuity theorem allows for various probabilistic applications (limit theorems, support, large deviations, ...).

## 1. Introduction

Let us recall some basic ideas of (second order) viscosity theory [6, 7] and rough path theory [26, 27]. As for viscosity theory, consider a real-valued function $u = u(x)$ with $x \in \mathbb{R}^n$ and assume $u \in C^2$ is a classical supersolution,

$$-G\left(x, u, Du, D^2u\right) \geq 0,$$

where $G$ is a (continuous) function, *degenerate elliptic* in the sense that $G\left(x, u, p, A\right) \leq G\left(x, u, p, A + B\right)$ whenever $B \geq 0$ in the sense of symmetric matrices. The idea is to consider a (smooth) test function $\varphi$ which touches $u$ from below at some point $\bar{x}$. Basic calculus implies that $Du\left(\bar{x}\right) = D\varphi\left(\bar{x}\right)$ , $D^2u\left(\bar{x}\right) \geq D\varphi\left(\bar{x}\right)$ and, from degenerate ellipticity,

$$-G\left(\bar{x}, \varphi, D\varphi, D^2\varphi\right) \geq 0. \tag{1.1}$$

This suggests to define a *viscosity supersolution* (at the point $\bar{x}$) to $G = 0$ as a continuous function $u$ with the property that (1.1) holds for any test function which touches $u$ from below at $\bar{x}$. Similarly, *viscosity subsolutions* are defined via testfunctions touching $u$ from above and by reversing inequality in (1.1); *viscosity solutions* are both super- and subsolutions. Observe that this definition covers (completely degenerate) first order equations as well as parabolic equations, e.g. by considering $\partial_t - F = 0$ on $\mathbb{R}^+ \times \mathbb{R}^n$ where $F$ is degenerate elliptic. The resulting theory (existence, uniqueness, stability, ...) is without doubt one of most important recent developments in the field of partial differential equations. As a typical result, the initial value problem $\left(\partial_t - F\right) u = 0$, $u\left(0, \cdot\right) = u_0 \in \mathrm{BUC}\left(\mathbb{R}^n\right)$ has a unique solution in $\mathrm{BUC}\left(\left[0, T\right] \times \mathbb{R}^n\right)$ provided $F = F(t, x, Du, D^2u)$ is continuous, degenerate elliptic and satisfies a (well-known) technical condition (see condition 1 below). In fact,

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uniqueness follows from a stronger property known as *comparison:* assume $u$ (resp. $v$) is a supersolution (resp. subsolution) and $u_0 \geq v_0$; then $u \geq v$ on $[0,T] \times \mathbb{R}^n$. A key feature of viscosity theory is what workers in the field simply call *stability properties*. For instance, it is relatively straight-forward to study $(\partial_t - F)\, u = 0$ via a sequence of approximate problems, say $(\partial_t - F^n)\, u^n = 0$, provided $F^n \to F$ locally uniformly and some apriori information on the $u^n$ (e.g. locally uniform convergence, or locally uniform boundedness[1]. Note the stark contrast to the classical theory where ones has to control the actual deriviatives of $u^n$.

The idea of stability is also central to *rough path theory*. Given a collection $(V_1, \dots, V_d)$ of (sufficiently nice) vector fields on $\mathbb{R}^n$ and $z \in C^1\left([0,T], \mathbb{R}^d\right)$ one considers the (unique) solution $y$ to the ordinary differential equation

$$\dot{y}(t) = \sum_{i=1}^{d} V_i(y)\, \dot{z}^i(t), \quad y(0) = y_0 \in \mathbb{R}^n. \tag{1.2}$$

The question is, if the output signal $y$ depends in a stable way on the driving signal $z$. The answer, of course, depends strongly on how to measure distance between input signals. If one uses the $\infty$ norm, so that the distance between driving signals $z, \tilde{z}$ is given by $|z - \tilde{z}|_{\infty;[0,T]}$, then the solution will in general *not* depend continuously on the input.

**Example 1.** *Take* $n = 1, d = 2, V = (V_1, V_2) = (\sin(\cdot), \cos(\cdot))$ *and* $y_0 = 0$. *Obviously,*

$$z^n(t) = \left(\frac{1}{n}\cos\left(2\pi n^2 t\right), \frac{1}{n}\sin\left(2\pi n^2 t\right)\right)$$

*converges to* $0$ *in* $\infty$*-norm whereas the solutions to* $\dot{y}^n = V(y^n)\, \dot{z}^n, y_0^n = 0$, *do not converge to zero (the solution to the limiting equation* $\dot{y} = 0$*).*

If $|z - \tilde{z}|_{\infty;[0,T]}$ is replaced by the (much) stronger distance

$$|z - \tilde{z}|_{1\text{-var};[0,T]} = \sup_{(t_i) \subset [0,T]} \sum \left| z_{t_i,t_{i+1}} - \tilde{z}_{t_i,t_{i+1}} \right|,$$

it is elementary to see that now the solution map is continuous (in fact, locally Lipschitz); however, this continuity does not lend itself to push the meaning of (1.2): the closure of $C^1$ (or smooth) paths in variation is precisely $W^{1,1}$, the set of absolutely continuous paths (and thus still far from a typical Brownian path). Lyons' theory of rough paths exhibits an entire cascade of ($p$-variation or $1/p$-Hölder type rough path) metrics, for each $p \in [1, \infty)$, on path-space under which such ODE solutions are continuous (and even locally Lipschitz) functions of their driving signal. For instance, the "rough path" $p$-variation distance between two smooth $\mathbb{R}^d$-valued paths $z, \tilde{z}$ is given by

$$\max_{j=1,\dots,[p]} \left( \sup_{(t_i) \subset [0,T]} \sum \left| z^{(j)}_{t_i,t_{i+1}} - \tilde{z}^{(j)}_{t_i,t_{i+1}} \right|^p \right)^{1/p}$$

where $z^{(j)}_{s,t} = \int dz_{r_1} \otimes \cdots \otimes dz_{r_j}$ with integration over the $j$-dimensional simplex $\{s < r_1 < \cdots < r_j < t\}$. This allows to extend the very meaning of (1.2), in a unique

[1] What we have in mind here is the *Barles–Perthame method of semi-relaxed limits*. We shall use this method in the proof of theorem 1 and postpone precise references until then.

and continuous fashion, to driving signals which live in the abstract completion of smooth $\mathbb{R}^d$-valued paths (with respect to rough path $p$-variation or a similarly defined $1/p$-Hölder metric). The space of socalled $p$-rough paths[2] is precisely this abstract completion. In fact, this space can be realized as genuine path space,

$$C^{0,p\text{-var}}\left([0,T],G^{[p]}\left(\mathbb{R}^d\right)\right) \text{ resp. } C^{0,1/p\text{-Höl}}\left([0,T],G^{[p]}\left(\mathbb{R}^d\right)\right)$$

where $G^{[p]}\left(\mathbb{R}^d\right)$ is the free step-$[p]$ nilpotent group over $\mathbb{R}^d$, equipped with Carnot–Caratheodory metric; realized as a subset of $1+\mathfrak{t}^{[p]}\left(\mathbb{R}^d\right)$ where

$$\mathfrak{t}^{[p]}\left(\mathbb{R}^d\right)=\mathbb{R}^d\oplus\left(\mathbb{R}^d\right)^{\otimes 2}\oplus\cdots\oplus\left(\mathbb{R}^d\right)^{\otimes [p]}$$

is the natural statespace for (up to $[p]$) iterated integrals of a smooth $\mathbb{R}^d$-valued path. For instance, almost every realization of $d$-dimensional Brownian motion $B$ *enhanced with its iterated stochastic integrals in the sense of Stratonovich,* i.e. the matrix-valued process given by

$$B^{(2)} := \left(\int_0^{\cdot} B^i \circ dB^j\right)_{i,j\in\{1,\dots,d\}} \tag{1.3}$$

yields a path $\mathbf{B}(\omega)$ in $G^2\left(\mathbb{R}^d\right)$ with finite $1/p$-Hölder (and hence finite $p$-variation) regularity, for any $p>2$. ($\mathbf{B}$ is known as *Brownian rough path.*) We remark that $B^{(2)}=\frac{1}{2}B\otimes B+A$ where $A:=\operatorname{Anti}\left(B^{(2)}\right)$ is known as *Lévy's stochastic area;* in other words $\mathbf{B}(\omega)$ is determined by $(B,A)$, i.e. Brownian motion *enhanced with Lévy's area.*

Turning to the main topic of this paper, we follow [21, 22, 23] in considering a real-valued function of time and space $u=u(t,x)\in \mathrm{BUC}([0,T]\times\mathbb{R}^n)$ which solves the nonlinear partial differential equation

$$\begin{aligned} du &= F\left(t,x,Du,D^2u\right)dt+\sum_{i=1}^d H_i(x,Du)\,dz^i \\ &\equiv F\left(t,x,Du,D^2u\right)dt+H(x,Du)\,dz \end{aligned} \tag{1.4}$$

in viscosity sense. When $z:[0,T]\to\mathbb{R}^d$ is $C^1$ then, subject to suitable conditions on $F$ and $H$, this falls in the standard setting of viscosity theory as discussed above. This can be pushed further to $z\in W^{1,1}$ (see e.g. [21, Remark 4] and the references given there) but the case when $z=z(t)$ has only "Brownian" regularity (just below 1/2-Hölder, say) falls dramatically outside the scope of the standard theory. The reader can find a variety of examples (drawing from fields as diverse as stochastic control theory, pathwise stochastic control, interest rate theory, front propagation and phase transition in random media, ...) in the articles [22, 20] justifying the need of a theory of (non-linear) *stochastic partial differential equations* (SPDEs) in which $z$ in (1.4) is taken as a Brownian motion[3]. In the same series of articles a satisfactory theory is established for the case of non-linear Hamiltonian with no spatial dependence, i.e. $H=H(Du)$. The contribution of this article is to deal with non-linear $F$ and $H=H(x,Du)$, linear in $Du$, although we suspect that the marriage of rough path and viscosity methodology will also prove useful in further investigations on *fully nonlinear* (i.e. both $F$ and $H$) stochastic partial differential

[2]In the strict terminology of rough path theory: geometric $p$-rough paths.

[3]... in which case (1.4) is understood in Stratonovich form.

equations[4]. To fix ideas, we give the following example, suggested in [22] and carefully worked out in [3, 4].

**Example 2** (Pathwise stochastic control)**.** *Consider*

$$dX = b(X;\alpha)\,dt + W(X;\alpha)\circ d\tilde{B} + V(X)\circ dB,$$

*where $b, W, V$ are (collections of) sufficiently nice vector fields (with $b, W$ dependent on a suitable control $\alpha = \alpha(t) \in \mathcal{A}$, applied at time $t$) and $\tilde{B}, B$ are multi-dimensional (independent) Brownian motions. Define*

$$v(x,t;B) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\left(g\left(X_T^{x,t}\right) + \int_t^T f\left(X_s^{x,t},\alpha_s\right)ds\right)\middle| B\right]$$

*where $X^{x,t}$ denotes the solution process to the above SDE started at $X(t) = x$. Then, at least by a formal computation,*

$$dv + \inf_{\alpha\in\mathcal{A}}\left[b(x,\alpha)Dv + L_a v + f(x,\alpha)\right]dt + Dv\cdot V(x)\circ dB = 0$$

*with terminal data $v(\cdot,T)\equiv g$, and $L_\alpha = \sum W_i^2$ in Hörmander form. Setting $u(x,t) = v(x,T-t)$ turns this into the initial value (Cauchy) problem,*

$$du = \inf_{\alpha\in\mathcal{A}}\left[b(x,\alpha)Du + L_a u + f(x,\alpha)\right]dt + Du\cdot V(x)\circ dB_{T-}.$$

*with initial data $u(\cdot,0)\equiv g$; and hence of a form which is covered by theorem 1 below. Indeed, $H = (H_1, H_2)$, $H_i(x,p) = p\cdot V_i(x)$, is linear in $p$. (Moreover, the rough driving signal in theorem 1 is taken as $\mathbf{z}_t := \mathbf{B}_{T-t}(\omega)$ where $\mathbf{B}(\omega)$ is a fixed Brownian rough path, run backwards in time.[5])*

Returning to the general setup of (1.4), the results [21, 22, 23] are in fact *pathwise* and apply to any continuous path $z\in C\left([0,T],\mathbb{R}^d\right)$, this includes Brownian and even rougher sources of noise; however, the assumption was made that $H = H(Du)$ is independent of $x$. The rôle of $x$-dependence is an important one (as it arises in applications such as example 2): the results of Lions–Souganidis imply that the map

$$z\in C^1\left([0,T],\mathbb{R}^d\right)\mapsto u(\cdot,\cdot)\in C\left([0,T],\mathbb{R}^n\right)$$

depends continuously on $z$ *in uniform topology*; thereby giving existence/uniqueness results to

$$du = F\left(t,x,Du,D^2u\right)dt + \sum_{i=1}^d H_i(Du)\,dz^i$$

for *every* continuous path $z:[0,T]\to\mathbb{R}^d$. When the Hamiltonian depends on $x$, this ceases to be true; indeed, take $F\equiv 0, d=2$ and $H_i(x,p) = pV_i(x)$ where $V_1, V_2$ are the vector fields from example 1. Solving the characteristic equations shows that $u$ is expressed in terms of the (inverse) flow associated to $dy = V_1(y)\,dz^1 + V_2(y)\,dz^2$, and we have already seen that the solution of this ODE does not depend continuously on $z = \left(z^1,z^2\right)$ in uniform topology[6].

[4]The use of rough path analysis in the context of nonlinear SPDEs was verbally conjectured by P.L. Lions in his 2003 Courant lecture.

[5]Alternatively, the proof of theorem 1 is trivially modified to directly accomodate terminal data problems.

[6]We shall push this remark much further in theorem 2 below.

Of course, this type of problem can be prevented by strengthening the topology: the Lyons-theory of rough paths does exhibit an entire cascade of ($p$-variation or $1/p$-Hölder type rough path) metrics (for each $p \geq 1$) on path-space under which such ODE solutions are continuous functions of their driving signal. This suggests to extend the Lions–Souganidis theory from a pathwise to a *rough* pathwise theory. We shall do so for a rich class of fully-nonlinear $F$ and Hamiltonians $H(x, Du)$ linear in $Du$. This last assumption allows for a global change of coordinates which mimicks a classical trick in SPDE analysis (Tubaro, Kunita, ...) where a SPDE is transformed into a random PDE (i.e. one that can be solved with deterministic methods by fixing randomness). In doing so, the interplay between rough path and viscosity methods is illustrated in a transpart way and everything boils down to combine the stability properties of viscosity solution with those of differential equations in the rough path sense. We have the following result.[7]

**Theorem 1.** *Let $p \geq 1$ and $(z^{\varepsilon}) \subset C^{\infty}\left([0,T], \mathbb{R}^d\right)$ be Cauchy in (p-variation) rough path topology with rough path limit $\mathbf{z} \in C^{0,p\text{-}var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$. Assume*

$$u_0^{\varepsilon} \in \mathrm{BUC}\left(\mathbb{R}^n\right) \to u_0 \in \mathrm{BUC}\left(\mathbb{R}^n\right),$$

*locally uniformly as $\varepsilon \to 0$. Let $F = F(t, x, p, X)$ be continuous, degenerate elliptic, and assume that $\partial_t - F$ satisfies $\Phi^{(3)}$-invariant comparison (cf. definition 1 below). Assume that $V = (V_1, \ldots, V_d)$ is a collection of $\mathrm{Lip}^{\gamma+2}\left(\mathbb{R}^n; \mathbb{R}^n\right)$ vector fields with $\gamma > p$. Consider (necessarily unique[8]) viscosity solutions $u^{\varepsilon} \in \mathrm{BUC}\left([0,T] \times \mathbb{R}^n\right)$ to*

$$du^{\varepsilon} = F\left(t, x, Du^{\varepsilon}, D^2 u^{\varepsilon}\right) dt - Du^{\varepsilon}(t,x) \cdot V(x)\, dz^{\varepsilon}(t), \tag{1.5}$$

$$u^{\varepsilon}(0, \cdot) = u_0^{\varepsilon} \tag{1.6}$$

*and assume that the resulting family $(u^{\varepsilon} : \varepsilon > 0)$ is locally uniformly bounded[9]. Then (i) there exists a unique $u \in \mathrm{BUC}\left([0,T] \times \mathbb{R}^n\right)$, only dependent on $\mathbf{z}$ and $u_0$ but not on the particular approximating sequences, such that $u^{\varepsilon} \to u$ locally uniformly. We write (formally)*

$$du = F\left(t, x, Du, D^2 u\right) dt - Du(t,x) \cdot V(x)\, d\mathbf{z}(t), \tag{1.7}$$

$$u(0, \cdot) = u_0, \tag{1.8}$$

*and also $u = u^{\mathbf{z}}$ when we want to indicate the dependence on $\mathbf{z}$;*
*(ii) we have the contraction property*

$$\left|u^{\mathbf{z}} - \hat{u}^{\mathbf{z}}\right|_{\infty; \mathbb{R}^n \times [0,T]} \leq \left|u_0 - \hat{u}_0\right|_{\infty; \mathbb{R}^n}$$

*where $\hat{u}^{\mathbf{z}}$ is defined as limit of $\hat{u}^{\eta}$, defined as in (1.5) with $u^{\varepsilon}$ replaced by $\hat{u}^{\eta}$ throughout;*
*(iii) the solution map $(\mathbf{z}, u_0) \mapsto u^{\mathbf{z}}$ from*

$$C^{p\text{-}var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right) \times \mathrm{BUC}\left(\mathbb{R}^n\right) \to \mathrm{BUC}\left([0,T] \times \mathbb{R}^n\right)$$

*is continuous.*

[7] Unless otherwise stated we shall always equip $BUC$-spaces with the topology of locally uniform convergence.

[8] This follows from the first 5 lines in the proof of this theorem.

[9] A simple sufficient conditions is boundedness of $F(\cdot, \cdot, 0, 0)$ on $[0,T] \times \mathbb{R}^n$, and the assumption that $u_0^{\varepsilon} \to u_0$ uniformly, as can be seen by comparison.

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## 2. Condition for comparison

We shall always assume that $F = F(t,x,p,X)$ is continuous and degenerate elliptic. A sufficient condition[10] for comparison of BUC-solutions to $\partial_t = F$ on $[0,T] \times \mathbb{R}^n$ is given by

**Condition 1** ([6, (3.14)])**.** *There exists a function* $\theta : [0,\infty] \to [0,\infty]$ *with* $\theta(0+) = 0$*, such that for each fixed* $t \in [0,T]$*,*

$$F(t,x,\alpha(x-\tilde{x}),X) - F(t,\tilde{x},\alpha(x-\tilde{x}),Y) \leq \theta\left(\alpha|x-\tilde{x}|^2 + |x-\tilde{x}|\right) \tag{2.1}$$

*for all* $\alpha > 0$*,* $x,\tilde{x} \in \mathbb{R}^n$ *and* $X,Y \in S^n$ *(the space of* $n \times n$ *symmetric matrices) satisfy*

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{2.2}$$

*Furthermore, we require* $F = F(t,x,p,X)$ *to be uniformly continuous whenever* $p,X$ *remain bounded.*

Although this seems part of the folklore in viscosity theory[11] only the case when $\mathbb{R}^n$ is replaced by a bounded domain is discussed in detail in the literature ([6, (3.14) and Section 8] or [7, Section V.7, V.8]; in this case the very last requirement on uniform continuity can be omitted). For this reason and the reader's convenience we have included a full proof of parabolic comparison on $[0,T]\times\mathbb{R}^n$ under the above condition in the appendix.

**Remark 1** (Stability under sup, inf etc)**.** *Using the elementary inequalities,*

$$|\sup(.) - \sup(*)|, |\inf(.) - \inf(*)| \leq \sup|. - *|$$

*one immediately sees that if* $F_\gamma, F_{\gamma,\beta}$ *satisfy (2.1) for* $\gamma,\beta$ *in some index set with a common modulus* $\theta$*, then* $\inf_\gamma F_\gamma, \sup_\beta \inf_\gamma F_{\beta,\gamma}$ *etc again satisfy (2.1). Similar remarks apply to the uniform continuity property; provided there exists, for any* $R < \infty$*, a common modulus of continuity* $\sigma_R$*, valid whenever* $p,X$ *are of norm less than* $R$*.*

## 3. Invariant comparison

To motivate our key assumption on $F$ we need some preliminary remarks on the transformation behaviour of

$$Du = (\partial_1 u, \ldots, , \partial_n u), \ D^2 u = (\partial_{ij} u)_{i,j=1,\ldots,n}$$

under change of coordinates on $\mathbb{R}^n$ where $u = u(t,\cdot)$, for fixed $t$. Let us allow the change of coordinates to depend on $t$, say $v(t,\cdot) := u(t,\phi_t(\cdot))$ where $\phi_t : \mathbb{R}^n \to$

[10]... which actually implies degenerate ellipticity, cf. page 18 in [6, (3.14)].

[11]E.g. in Section 4.4. of Barles' 1997 lecture notes, www.phys.univ-tours.fr/~barles/Toulcours.pdf, or section V.9 in [7].

$\mathbb{R}^n$ is a diffeomorphism. Differentiating $v\left(t,\phi_t^{-1}(\cdot)\right) = u(t,\cdot)$ twice, followed by evaluation at $\phi_t(y)$, we have, with summation over repeated indices,

$$\begin{aligned}\partial_i u(t,\phi_t(x)) &= \partial_k v(t,x)\,\partial_i \phi_t^{-1;k}|_{\phi_t(x)}\\ \partial_{ij} u(t,\phi_t(x)) &= \partial_{kl} v(t,x)\,\partial_i \phi_t^{-1;k}|_{\phi_t(x)} \partial_j \phi_t^{-1;l}|_{\phi_t(x)} + \partial_k v(t,x)\,\partial_{ij}\phi_t^{-1;k}|_{\phi_t(x)}.\end{aligned}$$

We shall write this, somewhat imprecisely[12] but convenient, as

$$\begin{aligned}Du|_{\phi_t(x)} &= \left\langle Dv|_x, D\phi_t^{-1}|_{\phi_t(x)}\right\rangle,\\ D^2u|_{\phi_t(x)} &= \left\langle D^2v|_x, D\phi_t^{-1}|_{\phi_t(x)} \otimes D\phi_t^{-1}|_{\phi_t(x)}\right\rangle + \left\langle Dv|_x, D^2\phi_t^{-1}|_{\phi_t(x)}\right\rangle.\end{aligned} \tag{3.1}$$

Let us now introduce $\Phi^{(k)}$ as the class of all flows of $C^k$-diffeomorphisms of $\mathbb{R}^n$, $\phi = (\phi_t : t \in [0,T])$, such that $\phi_0 = \mathrm{Id}\ \forall \phi \in \Phi^{(k)}$ and such that $\phi_t$ and $\phi_t^{-1}$ have $k$ bounded derivatives, uniformly in $t \in [0,T]$. We say that $\phi(n) \to \phi$ in $\Phi^{(k)}$ iff for all multi-indices $\alpha$ with $|\alpha| \le k$

$$\partial_\alpha \phi(n) \to \partial_\alpha \phi_t,\ \partial_\alpha \phi(n)^{-1} \to \partial_\alpha \phi_t^{-1} \text{ locally uniformly in } [0,T]\times\mathbb{R}^n.$$

**Definition 1** ($\Phi^{(k)}$**-invariant comparison;** $F^\phi$)**.** *Let $k \ge 2$ and define $F^\phi((t,x,p,X))$ as*

$$F\left(t,\phi_t(x), \left\langle p, D\phi_t^{-1}|_{\phi_t(x)}\right\rangle, \left\langle X, D\phi_t^{-1}|_{\phi_t(x)} \otimes D\phi_t^{-1}|_{\phi_t(x)}\right\rangle + \left\langle p, D^2\phi_t^{-1}|_{\phi_t(x)}\right\rangle\right) \tag{3.2}$$

*We say that $\partial_t = F$ satisfies $\Phi^{(k)}$-invariant comparison if, for every $\phi \in \Phi^{(k)}$, comparison holds for BUC solutions of $\partial_t - F^\phi = 0$. More precisely, if $u$ is a sub- and $v$ a super-solution to this equation (in viscosity sense, both BUC)and $u(0,\cdot) \le v(0,\cdot)$ then $u \le v$ on $[0,T]\times\mathbb{R}^n$.*

## 4. Examples

**Example 3** ($F$ linear)**.** *Suppose that $\sigma(t,x) : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times n'}$ and $b(t,x) : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ are bounded, continuous in $t$ and Lipschitz continuous in $x$, uniformly in $t \in [0,T]$. If $F(t,x,p,X) = \mathrm{Tr}\left[\sigma(t,x)\,\sigma(t,x)^T X\right] + b(t,x)\cdot p$, then $\Phi^{(3)}$-invariant comparison holds. Although this is a special case of the following example, let us point out that $F^\phi$ is of the same form as $F$ with $\sigma, b$ replaced by*

$$\begin{aligned}\sigma^\phi(t,x)^k_m &= \sigma^i_m(t,\phi_t(x))\,\partial_i\phi_t^{-1;k}|_{\phi_t(x)},\ \ k=1,\dots,n; m=1,\dots,n'\\ b^\phi(t,x)^k &= \left[b^i(t,\phi_t(x))\,\partial_i\phi_t^{-1;k}|_{\phi_t(x)}\right] + \sum_{i,j}\left(\sigma^i_m\sigma^j_m\partial_{ij}\phi_t^{-1;k}|_{\phi_t(y)}\right),\ \ k=1,\dots,n.\end{aligned}$$

*By defining properties of flows of diffeomorphisms, $t \mapsto \partial_i\phi_t^{-1;k}|_{\phi_t(x)}, \partial_{ij}\phi_t^{-1;k}|_{\phi_t(y)}$ is continuous and the $C^3$-boundedness assumption inherent in our definition of $\Phi^{(3)}$ ensures that $\sigma^\phi, b^\phi$ are Lipschitz in $x$, uniformly in $t \in [0,T]$. It is then easy to see (cf. the argument of* [7, Lemma 7.1]*) that $F^\phi$ satisfiws condition 1 for every $\phi \in \Phi^{(3)}$. This implies that $\Phi^{(3)}$-invariant comparison holds for BUC solutions of $\partial_t - F^\phi = 0$.*

[12]Strictly speaking, one should view $\left(Du, D^2u\right)|.$ as *second order* cotangent vector, the pull-back of $\left(Dv, D^2v\right)|_x$ under $\phi_t^{-1}$.

**Example 4** ($F$ quasi-linear)**.** *Let*

$$F(t,x,p,X)=\operatorname{Tr}\left[\sigma(t,x,p)\,\sigma(t,x,p)^{T}X\right]+b(t,x,p)\,. \tag{4.1}$$

*We assume $b=b(t,x,p):[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is continuous, bounded and Lipschitz continuous in $x$ and $p$, uniformly in $t\in[0,T]$. We also assume that $\sigma=\sigma(t,x,p):[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times n'}$ is a continuous, bounded map such that*

- *$\sigma(t,\cdot,p)$ is Lipschitz continuous, uniformly in $(t,p)\in[0,T]\times\mathbb{R}^n$;*
- *there exists a constant $c_1>0$, such that*[13]

$$\forall p,q\in\mathbb{R}^n:|\sigma(t,x,p)-\sigma(t,x,q)|\le c_1\frac{|p-q|}{1+|p|+|q|} \tag{4.2}$$

*for all $t\in[0,T]$ and $x\in\mathbb{R}^n$.*

*We show that $F^{\phi}$ satisfies condition 1 for every $\phi\in\Phi^{(3)}$; this implies that $\Phi^{(3)}$-invariant comparison holds for $\partial_t=F$ with $F$ given by (4.1). To see this we proceed as follows. For brevity denote*

$$\begin{aligned} p &= \alpha(x-\tilde{x})\,,J_\cdot=D\phi_t^{-1}|_{\phi_t(\cdot)},H_\cdot=D^2\phi_t^{-1}|_{\phi_t(\cdot)}\\ \sigma_\cdot &= \sigma(t,\phi_t(\cdot),\langle p,J_\cdot\rangle)\,,a_\cdot=\sigma_\cdot\sigma_\cdot^T,\ b_\cdot=b(t,\phi_t(\cdot),\langle p,J_\cdot\rangle)\end{aligned}$$

*so that*

$$\begin{aligned} F^{\phi}(t,x,p,X) &= \operatorname{Tr}\left[a_x\left(\langle X,J_x\otimes J_x\rangle+\langle p,H_x\rangle\right)\right]+b_x\\ &= \operatorname{Tr}\left[J_xa_xJ_x^TX\right]+b_x+\operatorname{Tr}\left[a_x\langle p,H_x\rangle\right].\end{aligned}$$

*Hence*

$$F^{\phi}(t,\tilde{x},p,Y)-F^{\phi}(t,x,p,X)=\underbrace{\operatorname{Tr}\left[J_{\tilde{x}}a_{\tilde{x}}J_{\tilde{x}}^TY-J_xa_xJ_x^TX\right]}_{=:(i)}+\underbrace{b_{\tilde{x}}-b_x}_{=:(ii)}+\underbrace{\operatorname{Tr}\left[a_{\tilde{x}}\langle p,H_{\tilde{x}}\rangle-a_x\langle p,H_x\rangle\right]}_{=:(iii)}.$$

*To estimate $(i)$ note that $J_xa_xJ_x^T=J_x\sigma_x(J_x\sigma_x)^T$. The $\mathbb{R}^{2n}\times\mathbb{R}^{2n}$ matrix*

$$\begin{pmatrix}(J_x\sigma_x)(J_x\sigma_x)^T & J_x\sigma_x(J_{\tilde{x}}\sigma_{\tilde{x}})^T\\ (J_{\tilde{x}}\sigma_{\tilde{x}})(J_x\sigma_x)^T & J_{\tilde{x}}\sigma_{\tilde{x}}(J_{\tilde{x}}\sigma_{\tilde{x}})^T\end{pmatrix}$$

*is positive semidefinite and thus we can multiply it to both sides of the inequality*

$$\begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix}\le 3\alpha\begin{pmatrix}I & -I\\ -I & I\end{pmatrix}.$$

*The resulting inequality is stable under evaluating the trace and so one gets*

$$\begin{aligned}\operatorname{Tr}\left[J_x\sigma_x(J_x\sigma_x)^T\cdot X-J_{\tilde{x}}\sigma_{\tilde{x}}(J_{\tilde{x}}\sigma_{\tilde{x}})^T\cdot Y\right] &\le 3\alpha\operatorname{Tr}\Big[(J_x\sigma_x)(J_x\sigma_x)^T-J_x\sigma_x(J_{\tilde{x}}\sigma_{\tilde{x}})^T\\ &\qquad -J_{\tilde{x}}\sigma_{\tilde{x}}(J_x\sigma_x)^T+J_{\tilde{x}}\sigma_{\tilde{x}}(J_{\tilde{x}}\sigma_{\tilde{x}})^T\Big]\\ &= 3\alpha\operatorname{Tr}\left[(J_x\sigma_x-J_{\tilde{x}}\sigma_{\tilde{x}})(J_x\sigma_x-J_{\tilde{x}}\sigma_{\tilde{x}})^T\right]\\ &= 3\alpha\,\|J_x\sigma_x-J_{\tilde{x}}\sigma_{\tilde{x}}\|^2\end{aligned}$$

*(using that $\operatorname{Tr}\left[\cdot\,\cdot^T\right]$ defines an inner product for matrices and gives rise to the Frobenius matrix norm $\|.\|$). Hence, by the triangle inequality and Lipschitzness*

[13] A condition of this type also appears also in [1].

*of the Jacobian of the flow (which follows a fortiori from the boundedness of the second order derivatives of the flow),*

$$\begin{aligned}\|J_x\sigma_x - J_{\tilde{x}}\sigma_{\tilde{x}}\| &\leq \|J_x\sigma_x - J_x\sigma_{\tilde{x}}\| + \|J_x\sigma_{\tilde{x}} - J_{\tilde{x}}\sigma_{\tilde{x}}\| \\ &\leq \|J_x\| \|\sigma_x - \sigma_{\tilde{x}}\| + \|J_x - J_{\tilde{x}}\| \|\sigma_{\tilde{x}}\| \\ &\leq \|J_x\| \|\sigma_x - \sigma_{\tilde{x}}\| + c_2(\sigma,\phi)|x-\tilde{x}|\end{aligned}$$

*Since $\sigma(t,\cdot,q)$ is Lipschitz continuous (uniformly in $(t,q)\in[0,T]\times\mathbb{R}^n$) and $\phi_t(\cdot)$ is Lipschitz continuous (uniformly in $t\in[0,T]$), we can use our assumption (4.2) on $\sigma$, to see*

$$\|\sigma_x - \sigma_{\tilde{x}}\| \leq (const) \times |x-\tilde{x}|. \tag{4.3}$$

*Indeed,*

$$\begin{aligned}\|\sigma_x - \sigma_{\tilde{x}}\| &= \|\sigma(t,\phi_t(x),p\cdot J_x) - \sigma(t,\phi_t(\tilde{x}),p\cdot J_{\tilde{x}})\| \\ &\leq \|\sigma(t,\phi_t(x),p\cdot J_x) - \sigma(t,\phi_t(\tilde{x}),p\cdot J_x)\| \\ &\quad + \|\sigma(t,\phi_t(\tilde{x}),p\cdot J_x) - \sigma(t,\phi_t(\tilde{x}),p\cdot J_{\tilde{x}})\| \\ &\leq c_2(\sigma,\phi)|x-\tilde{x}| + c_1\frac{\alpha|x-\tilde{x}||J_x - J_{\tilde{x}}|}{1+\alpha|x-\tilde{x}|(|J_x|+|J_{\tilde{x}}|)};\end{aligned}$$

*and, noting that $\phi_t\circ\phi_t^{-1} = Id$ and $\sup_{(t,x)\in[0,T]\times\mathbb{R}^n}\|D\phi_t|_x\| \leq c_3$ implies $\|J_x\| = \left\|D\phi_t^{-1}|_{\phi_t(x)}\right\| \geq 1/c_3$, we have*

$$\begin{aligned}c_1\frac{\alpha|x-\tilde{x}||J_x - J_{\tilde{x}}|}{1+\alpha|x-\tilde{x}|(|J_x|+|J_{\tilde{x}}|)} &\leq |x-\tilde{x}|\cdot\frac{c_1\alpha|J_x - J_{\tilde{x}}|}{\alpha|x-\tilde{x}|(|J_x|+|J_{\tilde{x}}|)} \\ &\leq |x-\tilde{x}|\frac{c_4(\sigma,\phi)|x-\tilde{x}|}{|x-\tilde{x}|(|J_x|+|J_{\tilde{x}}|)} \\ &\leq c_5(\sigma,\phi)|x-\tilde{x}|.\end{aligned}$$

*Putting things together we have*

$$|(i)| \leq c_6(\sigma,\phi)\,\alpha|x-\tilde{x}|^2.$$

*As for $(ii)$, we have that,*

$$\begin{aligned}|b_x - b_{\tilde{x}}| &\leq |b(t,\phi_t(x),\langle p,J_x\rangle) - b(t,\phi_t(\tilde{x}),\langle p,J_x\rangle)| \\ &\quad + |b(t,\phi_t(\tilde{x}),\langle p,J_x\rangle) - b(t,\phi_t(\tilde{x}),\langle p,J_{\tilde{x}}\rangle)| \\ &\leq c_7(b)(|\phi_t(x)-\phi_t(\tilde{x})| + |p||J_{\tilde{x}} - J_x|)\end{aligned}$$

*where $c_7(b)$ is the (uniform in $t\in[0,T]$) Lipschitz bound for $b(t,\cdot,\cdot)$. To get the required estimate we again use the regularity of the flow. Finally, for $(iii)$,*

$$\begin{aligned}(iii) &= \mathrm{Tr}\left[a_{\tilde{x}}\langle p,H_{\tilde{x}}\rangle - a_{\tilde{x}}\langle p,H_x\rangle\right] + \mathrm{Tr}\left[a_{\tilde{x}}\langle p,H_x\rangle - a_x\langle p,H_x\rangle\right] \\ &= \mathrm{Tr}\left[a_{\tilde{x}}\langle p,H_{\tilde{x}} - H_x\rangle\right] + \mathrm{Tr}\left[(a_{\tilde{x}} - a_x)\langle p,H_x\rangle\right].\end{aligned}$$

*Using Cauchy-Schwartz (with inner product $\mathrm{Tr}\left[\cdot\,\cdot^T\right]$) and $p = \alpha(x-\tilde{x})$ it is clear that boundedness of $H$ and $a$ (i.e. $\sup_x|H_x| < \infty$ uniformly in $t\in[0,T]$ and similarly for $a$) and Lipschitz continuity (i.e.$|H_x - H_{\tilde{x}}| \leq (const)\times|x-\tilde{x}|$ uniformly in $t\in[0,T]$ and similar for $a$) will suffice to obtain the (desired) estimate*

$$|(iii)| \leq c_8\times\alpha|x-\tilde{x}|^2.$$

*Only Lipschitz continuity of $a_x = \sigma_x\sigma_x^T$ requires a discsussion. But this follows, thanks to boundedness of $\sup_x|\sigma_x|$, from showing Lipschitzness of $x\mapsto\sigma_x = \sigma(t,\phi_t(x),\langle p,J_x\rangle)$ uniformly in $t\in[0,T]$ which was already seen in (4.3). This*

*shows that $F^{\phi}$ satisfies (2.1), for any $\phi \in \Phi^{(3)}$. To see that $F^{\phi}$ satisfies condition 1 it only remains to see that $F^{\phi}(t,x,p,X)$ is uniformly continuous whenever $p, X$ remain bounded. To see this first observe that the flow map $\phi_t(x)$, as function of $(t,x) \in [0,T] \times \mathbb{R}^n$, is uniformly continuous (but not bounded) while the derivatives of the (inverse) flow, given by $J., H.$ above, are bounded uniformly continuous maps as functions of $t, x$. One now easily concludes with the fact the observations that (a) the product of BUC function is again BUC and (b) the composition of a BUC function with a UC function is again BUC.*

**Example 5** ($F$ of Hamilton-Jacobi-Bellman type)**.** *From the above examples and remark 1, we see that $\Phi^{(3)}$-invariant comparison holds when $F$ is given by*

$$F(t,x,p,X) = \inf_{\gamma \in \Gamma} \left\{ \mathrm{Tr}\left[ \sigma(t,x;\gamma)\,\sigma(t,x;\gamma)^T X \right] + b(t,x;\gamma)\cdot p \right\},$$

*the usual non-linearity in the Hamilton-Jacobi-Bellman equation, and more generally*

$$F(t,x,p,X) = \inf_{\gamma \in \Gamma} \left\{ \mathrm{Tr}\left[ \sigma(t,x,p;\gamma)\,\sigma(t,x,p;\gamma)^T \cdot X \right] + b(t,x,p;\tau) \right\}$$

*whenever the conditions in examples 3 and 4 are satisfied uniformly with respect to $\gamma \in \Gamma$.*

**Example 6** ($F$ of Isaac type)**.** *Similarly, $\Phi^{(3)}$-invariant comparison holds for*

$$F(t,x,p,X) = \sup_{\beta} \inf_{\gamma} \left\{ \mathrm{Tr}\left[ \sigma(t,x;\beta,\gamma)\,\sigma(t,x;\beta,\gamma)^T X \right] + b(t,x;\beta,\gamma)\cdot p \right\},$$

*(such non-linearities arise in Isaac equation in the theory of differential games), and more generally*

$$F(t,x,p,X) = \sup_{\beta} \inf_{\gamma} \left\{ \mathrm{Tr}\left[ \sigma(t,x,p;\beta,\gamma)\,\sigma(t,x,p;\beta,\gamma)^T \cdot X \right] + b(t,x,p;\beta,\gamma) \right\}$$

*whenever the conditions in examples 3 and 4 are satisfied uniformly with respect to $\beta \in \mathcal{B}$ and $\gamma \in \Gamma$, where $\mathcal{B}$ and $\Gamma$ are arbitrary index sets.*

## 5. Some lemmas

**Lemma 1.** *Let $z : [0,T] \to \mathbb{R}^d$ be smooth and assume that we are given $C^3$-bounded vector fields*[14] *$V = (V_1, \dots, V_d)$. Then ODE*

$$dy_t = V(y_t)\,dz_t, \quad t \in [0,T]$$

*has a unique solution flow (of $C^3$-diffeomorphisms) $\phi = \phi^z \in \Phi^{(3)}$.*

*Proof.* Standard, e.g. chapter 4 in [13]. ☐

**Proposition 1.** *Let $z, V$ and $\phi$ be as in lemma 1. Then $u$ is a viscosity sub- (resp. super-) solution (always assumed* BUC*) of*

$$\dot{u}(t,x) = F\left(t,x,Du,D^2u\right) - Du(t,x)\cdot V(x)\,\dot{z}(t) \tag{5.1}$$

*if and only if $v(t,x) := u(t,\phi_t(x))$ is a viscosity sub- (resp. super-) solution of*

$$\dot{v}(t,x) = F^{\phi}\left(t,x,Dv,D^2v\right) \tag{5.2}$$

*where $F^{\phi}$ was defined in (3.2).*

[14] In particular, if the vector fields are $\mathrm{Lip}^{\gamma}$, $\gamma > p+2$, $p \geq 1$, then they are also $C^3$-bounded.

*Proof.* Set $y = \phi_t(x)$. When $u$ is a classical sub-solution, it suffices to use the the chain-rule and definition of $F^\phi$ to see that

$$\begin{aligned}\dot{v}(t,x) &= \dot{u}(t,y) + Du(t,y)\cdot\dot{\phi}_t(x) = \dot{u}(t,y) + Du(t,y)\cdot V(y)\dot{z}_t \\ &\leq F\left(t,y,Du(t,y),D^2u(t,y)\right) = F^\phi\left(t,x,Dv(t,x),D^2v(t,x)\right).\end{aligned}$$

The case when $u$ is a viscosity sub-solution of (5.1) is not much harder: suppose that $(\bar{t},\bar{x})$ is a maximum of $v-\xi$, where $\xi \in C^2([0,T]\times\mathbb{R}^n)$ and define $\psi \in C^2([0,T]\times\mathbb{R}^n)$ by $\psi(t,y) = \xi\left(t,\phi_t^{-1}(y)\right)$. Set $\bar{y} = \phi_{\bar{t}}(\bar{x})$ so that

$$F\left(\bar{t},\bar{y},D\psi(\bar{t},\bar{y}),D^2\psi(\bar{t},\bar{y})\right) = F^\phi\left(\bar{t},\bar{x},D\xi(\bar{t},\bar{x}),D^2\xi(\bar{t},\bar{x})\right).$$

Obviously, $(\bar{t},\bar{y})$ is a maximum of $u-\psi$, and since $u$ is a viscosity sub-solution of (5.1) we have

$$\dot{\psi}(\bar{t},\bar{y}) + D\psi(\bar{t},\bar{y})V(\bar{y})\dot{z}(\bar{t}) \leq F\left(\bar{t},\bar{y},D\psi(\bar{t},\bar{y}),D^2\psi(\bar{t},\bar{y})\right).$$

On the other hand, $\xi(t,x) = \psi(t,\phi_t(x))$ implies $\dot{\xi}(\bar{t},\bar{x}) = \dot{\psi}(\bar{t},\bar{y}) + D\psi(\bar{t},\bar{y})V(\bar{y})\dot{z}(\bar{t})$ and putting things together we see that

$$\dot{\xi}(\bar{t},\bar{x}) \leq F^\phi\left(\bar{t},\bar{x},D\xi(\bar{t},\bar{x}),D^2\xi(\bar{t},\bar{x})\right)$$

which says precisely that $v$ is a viscosity sub-solution of (5.2). Replacing maximum by minimum and $\leq$ by $\geq$ in the preceding argument, we see that if $u$ is a super-solution of (5.1), then $v$ is a super-solution of (5.2).
Conversely, the same arguments show that if $v$ is a viscosity sub- (resp. super-) solution for (5.2), then $u(t,y) = v\left(t,\phi^{-1}(y)\right)$ is a sub- (resp. super-) solution for (5.1). □

## 6. Proof of the main result

*Proof.* **(Theorem 1.)** Using Lemma 1, we see that $\phi^\varepsilon \equiv \phi^{z^\varepsilon}$, the solution flow to $dy = V(y)\,dz^\varepsilon$, is an element of $\Phi \equiv \Phi^{(3)}$. Set $F^\varepsilon := F^{\phi^\varepsilon}$. From Proposition 1, we know that $u^\varepsilon$ is a solution to

$$du^\varepsilon = F\left(t,y,Du^\varepsilon,D^2u^\varepsilon\right)dt - Du^\varepsilon(t,y)\cdot V(y)\,dz^\varepsilon(t), \quad u^\varepsilon(0,\cdot) = u_0^\varepsilon$$

if and only if $v^\varepsilon$ is a solution to $\partial_t - F^\varepsilon = 0$. By assumption of $\Phi$-invariant comparison,

$$|v^\varepsilon - \hat{v}^\varepsilon|_{\infty;\mathbb{R}^n\times[0,T]} \leq |v_0 - \hat{v}_0|_{\infty;\mathbb{R}^n}.$$

where $v^\varepsilon, \hat{v}^\varepsilon$ are viscosity solution to $\partial_t - F^\varepsilon = 0$. Let $\phi^{\mathbf{z}}$ denote the solution flow to the rough differential equation

$$dy = V(y)\,d\mathbf{z}.$$

Thanks to $\mathrm{Lip}^{\gamma+2}$-regularity of the vector fields $\phi^{\mathbf{z}} \in \Phi$, and in particular a flow of $C^3$-diffeomorphisms. Set $F^{\mathbf{z}} = F^{\phi^{\mathbf{z}}}$. The "universal" limit theorem [26] holds, in fact, on the level of flows of diffeomorphisms (see [25] and [13, Chapter 11] for more details) tells us that, since $z^\varepsilon$ tends to $\mathbf{z}$ in rough path sense,

$$\phi^\varepsilon \to \phi^{\mathbf{z}} \text{ in } \Phi$$

so that, by continuity of $F$ (more precisely: uniform continuity on compacts), we easily deduce that

$$F^\varepsilon \to F^{\mathbf{z}} \text{ locally uniformly.}$$

From the method of semi-relaxed limits (Lemma 6.1 and Remarks 6.2, 6.3 and 6.4 in [6], see also [7]) the pointwise (relaxed) limits

$$\begin{aligned}\bar{v} \quad &: \quad = \limsup{}^{*} \ v^{\varepsilon}, \\ \underline{v} \quad &: \quad = \liminf{}_{*} \ v^{\varepsilon},\end{aligned}$$

are viscosity (sub resp. super) solutions to $\partial_t - F^{\mathbf{z}} = 0$, with identical initial data. As the latter equation satisfies comparison, one has trivially uniqueness and hence $v := \bar{v} = \underline{v}$ is the unique (and continuous, since $\bar{v}, \underline{v}$ are respectively upper resp. lower semi-continuous) solution to

$$\partial_t v = F^{\mathbf{z}} v \, , \ v(0, \cdot) = u_0(\cdot) \, .$$

Moreover, using a simple Dini-type argument (e.g. [6, p.35]) one sees that this limit must be uniform on compacts. It follows that $v$ is the unique solution to

$$\partial_t v = F^{\mathbf{z}} v \, , \ v(0, \cdot) = u_0(\cdot)$$

(hence does not depend on the approximating sequence to $\mathbf{z}$) and the proof of (i) is finished by setting

$$u^{\mathbf{z}}(t, x) := v\left(t, (\phi_t^{\mathbf{z}})^{-1}(x)\right).$$

(ii) The comparison $|u^{\mathbf{z}} - \hat{u}^{\mathbf{z}}|_{\infty;[0,T]\times\mathbb{R}^n} \leq |u_0 - \hat{u}_0|_{\infty;\mathbb{R}^n}$ is a simple consequence of comparison for $v, \hat{v}$ (solutions to $\partial_t v = F^{\mathbf{z}} v$). At last, to see (iii), we argue in the very same way as in (i), starting with

$$F^{\mathbf{z}_n} \to F^{\mathbf{z}} \text{ locally uniformly}$$

to see that $v^n \to v$ locally uniformly, i.e. uniformly on compacts. □

## 7. Applications to stochastic partial differential equations

Applications to SPDEs are path-by-path, i.e. by taking $\mathbf{z}$ to be a typical realization of Brownian motion and Lévy's area, $\mathbf{B}(\omega) \equiv (B, A)$, also known as enhanced Brownian motion or Brownian rough path. The continuity property (iii) of our theorem 1 easily allows to identify (1.7) with $\mathbf{z} = \mathbf{B}(\omega)$ as Stratonovich solution to the non-linear SPDE

$$du = F\left(t, x, Du, D^2 u\right) dt - Du(t, x) \cdot V(x) \circ dB, \ \ u(0, \cdot) = u_0.$$

Indeed, under the stated assumptions the Wong-Zakai approximations, in which the Brownian $B$ is replaced by its piecewise linear approximation, based on some mesh $\left\{0, \frac{T}{n}, \frac{2T}{n} \dots, T\right\}$, the approximate solution will convergence (locally uniformly on $[0, T] \times \mathbb{R}^n$ and in probability) to the solution of

$$du = F\left(t, x, Du, D^2 u\right) dt - Du(t, x) \cdot V(x) \, d\mathbf{B}, \ \ u(0, \cdot) = u_0,$$

as constructed in theorem 1. Let us give some applications, typical in the sense that they have been studied in great detail in the case of classical stochastic differential equations.

**(Approximations)** *Any* approximation result to $\mathbf{B}$ in rough path topology implies a corresponding (weak or strong) limit theorem for such SPDEs: it suffices that an approximation to $B$ converges in rough path topology; as is well known (e.g. [13, Chapter 13] and the references therein) examples include piecewise linear -, mollifier - and Karhunen-Loeve approximations, as well as (weak) Donsker type random walk approximations [2]. The point being made, we shall not spell out more details here.

**(Twisted approximations)**

The following results implies en pasant that there is no (classical) pathwise theory of SPDEs in presence of spatial dependence in the Hamilonian terms.

**Theorem 2.** *Let $V = (V_1, \ldots, V_d)$ be a collection of $C^\infty$-bounded vector fields on $\mathbb{R}^n$ and $B$ a $d$-dimensional standard Brownian motion. Then, for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \{1, \ldots, d\}^N$, $N \geq 2$, there exists (piecewise) smooth approximations $(z^k)$ to $B$, with each $z^k$ only dependent on $\{B(t) : t \in D^k\}$ where $(D^k)$ is a sequence of dissections of $[0,T]$ with mesh tending to zero, such that almost surely*

$$z^k \to B \text{ uniformly on } [0,T],$$

*but $u^k$, solutions to*

$$du^k = F\left(t, x, Du^k, D^2u^k\right) dt - Du^k(t,x) \cdot V(x)\, dz^k, \;\; u^k(0,\cdot) = u_0 \in \mathrm{BUC}(\mathbb{R}^n),$$

*(with assumptions on $F$ as formulated in theorem 1) converge almost surely locally uniformly to the solution of the "wrong" differential equation*

$$du = \left[F\left(t, x, Du, D^2u\right) - Du(t,x) \cdot V_\alpha(x)\right] dt - Du(t,x) \cdot V(x) \circ dB$$

*where $V_\alpha$ is the bracket-vector field given by $V_\alpha = \left[V_{\alpha_1}, \left[V_{a_2}, \ldots \left[V_{\alpha_{N-1}}, V_{\alpha_N}\right]\right]\right]$.*

*Proof.* The rough path regularity of $\mathbf{B}(\omega)$ implies that higher iterated (Stratonvich) integrals are deterministically defined; see [24, First thm.]. Doing this up to level $N$ yields a (rough path) $S_N(\mathbf{B})$ and we perturbe it in the highest level, linearly in the

$$\left[e_{\alpha_1}, \left[e_{a_2}, \ldots \left[e_{\alpha_{N-1}}, e_{\alpha_N}\right]\right]\right]\text{-direction}$$

of $S_N(\mathbf{B})$ viewed as element in the step-$N$ free nilpotent Lie algebra. This yields a (level-$N$) rough path $\tilde{\mathbf{B}}$ and we can find approximations $(z^k)$ that converge almost surely to $\tilde{\mathbf{B}}$ in rough path topology (see [9]). One identifies standard RDEs driven by $\tilde{\mathbf{B}}$ as RDEs-with-drift (driven along the original vector fields by $d\mathbf{B}$, and along $V_\alpha$ by $dt$). The resulting identification obviously holds on the level of RDE flows and thus

$$u^{z^k}(t,x) = v\left(t, \left(\phi_t^{z^k}\right)^{-1}(x)\right) \to u^{\tilde{\mathbf{B}}}(t,x) = v\left(t, \left(\phi_t^{\tilde{\mathbf{B}}}\right)^{-1}(x)\right)$$

The flow identification then implies that

$$du = F\left(t, x, Du, D^2u\right) dt - Du(t,x) \cdot V(x)\, d\tilde{\mathbf{B}}$$

is equivalent to the equation with $V(x)\, d\tilde{\mathbf{B}}$ replaced by $V(x)\, d\mathbf{B} + V_\alpha(x)\, dt$. □

**Remark 2.** *The attentive reader will have noticed that the preceding result also holds when the Stratonovich differential $\circ dB$ is replaced by $dz$ for some $z \in C^1\left([0,T], \mathbb{R}^d\right)$; it can then be viewed as result on the effective behaviour of a (deterministic) non-linear parabolic equations with coefficients that exhibit highly oscillatory behaviour in time.*

**(Support results)** In conjunction with known support properties of $\mathbf{B}$ (e.g. [19] in $p$-variation rough path topology or [8] for a conditional statement in Hölder rough path topology) continuity of the SPDE solution as a function of $\mathbf{B}$ immediately implies Stroock–Varadhan type support descriptions for such SPDEs. Let us note that, to the best of our knowledge, results of this type are new for such non-linear

SPDEs. In the linear case, approximations and support of SPDEs have been studied in great detail [18, 17, 15, 14, 16].

**(Large deviation results)** Another application of our continuity result is the ability to obtain large deviation estimates when $B$ is replaced by $\varepsilon B$ with $\varepsilon \to 0$; indeed, given the known large deiviation behaviour of $\left(\varepsilon B, \varepsilon^2 A\right)$ in rough path topology (e.g. [19] in $p$-variation and [11] in Hölder rough path topology) it suffices to recall that large deviation principles are stable under continuous maps. Again, large deviation estimates for non-linear SPDEs in the small noise limit appear to be new and may be hard to obtain without rough paths theory.

**(SPDEs with non-Brownian noise)** Yet another benefit of our approach is the ability to deal with SPDEs with non-Brownian and even non-semimartingale noise. For instance, one can take $\mathbf{z}$ as (the rough path lift of) fractional Brownian motion with Hurst parameter $1/4 < H < 1/2$, cf. [5] or [10], a regime which is "rougher" than Brownian and notoriously difficult to handle; or a diffusion with uniformly elliptic generator in divergence form with measurable coefficients; see [12]. Much of the above (approximations, support, large deviation) results also extend, as is clear from the respective results in the above-cited literature.

## 8. Appendix: comparison for parabolic equations

Let $u \in \mathrm{BUC}\left([0,T] \times \mathbb{R}^n\right)$ be a subsolution to $\partial_t - F$; that is, $\partial_t u - F\left(t, x, Du, D^2u\right) \leq 0$ if $u$ is smooth and with the usual viscosity definition otherwise. Similarly, let $v \in \mathrm{BUC}\left([0,T] \times \mathbb{R}^n\right)$ be a supersolution.

**Theorem 3.** *Assume condition 1. Then comparison holds. That is,*

$$u(0,\cdot) \leq v(0,\cdot) \implies u \leq v \text{ on } [0,T] \times \mathbb{R}^n.$$

*Proof.* We follow [6, Section 8]. Without loss of generality, we may assume that $\partial_t u - F\left(t, x, Du, D^2u\right) \leq -c < 0$ and that $\lim_{t\to T} u(t,x) = -\infty$ uniformly in $x \in R^n$. We aim to contradict the existence of a point $(s,z) \in (0,T) \times R^n$ such that

$$u(s,z) - v(s,z) = \delta > 0.$$

To this end, consider a maximum point $\left(\hat{t}, \hat{x}, \hat{y}\right) \in [0,T) \times R^n \times R^n$ of

$$\phi(t,x,y) = u(t,x) - v(t,y) - \frac{\alpha}{2}|x-y|^2 - \varepsilon\left(|x|^2 + |y|^2\right).$$

We first argue that, for small (resp. large) enough values of $\varepsilon$ and $\alpha$, the optimizing time paramter $\hat{t} \in [0,T)$ cannot be zero. Indeed, assuming $\hat{t} = 0$ we can estimate

$$\begin{aligned} \delta - 2\varepsilon|z|^2 &= \phi(s,z,z) \\ &\leq \phi(0,\hat{x},\hat{y}) \\ &= \sup_{x,y}\left[u_0(x) - v_0(y) - \frac{\alpha}{2}|x-y|^2 - \varepsilon\left(|x|^2+|y|^2\right)\right]; \end{aligned}$$

passing to the limit $\varepsilon \to 0, \alpha \to \infty$ on both sides (using lemma 2 below) then leaves us with the contradiction

$$0 < \delta \leq \sup_{x\in R^n}\left(u_0(x) - v_0(x)\right) \leq 0.$$

It follows that $\hat{t} \in (0,T)$. Again, the plan is to arrive at a contradiction (so that we have to reject the existence of a point $(s,z) \in (0,T) \times R^n$ at which $u(s,z) - v(s,z) > 0$) altogether. To this end, let us rewrite $\phi(t,x,y)$ as

$$\phi(t,x,y) = u^\varepsilon(t,x) - v^\varepsilon(t,y) - \frac{\alpha}{2}|x-y|^2$$

where $u^\varepsilon(t,x) = u(t,x) - \varepsilon|x|^2$ and $v^\varepsilon(t,y) = v(t,y) + \varepsilon|y|^2$. We may apply the (parabolic) theorem of sums [6, Thm 8.3] at $(\hat{t},\hat{x},\hat{y})$ to learn that there are numbers $a,b$ and $X,Y \in S^n$ such that

$$(a, \alpha(\hat{x}-\hat{y}), X) \in \bar{\mathcal{P}}^{2,+} u^\varepsilon(\hat{t},\hat{x}), \ (b, \alpha(\hat{x}-\hat{y}), Y) \in \bar{\mathcal{P}}^{2,-} v^\varepsilon(\hat{t},\hat{x}) \tag{8.1}$$

such that $a - b = 0$ and such that one has the estimate (2.2). It is easy to see (cf. [6, Remark 2.7]) that (8.1) is equivalent to

$$\begin{aligned} (a, \alpha(\hat{x}-\hat{y}) + 2\varepsilon\hat{x}, X + 2\varepsilon I) &\in \bar{\mathcal{P}}^{2,+} u(\hat{t},\hat{x}), \\ (b, \alpha(\hat{x}-\hat{y}) - 2\varepsilon\hat{y}, Y - 2\varepsilon I) &\in \bar{\mathcal{P}}^{2,-} v(\hat{t},\hat{x}); \end{aligned}$$

using that $\partial_t u - F(t,x,Du,D^2u) \le -c$ and $\partial_t v - F(t,x,Dv,D^2v) \ge 0$ (always understood in the sense of viscosity sub- resp. super-solutions) we then see that

$$\begin{aligned} a - F(\hat{t},\hat{x},\alpha(\hat{x}-\hat{y}) + 2\varepsilon\hat{x}, X + 2\varepsilon I) &\le -c, \\ b - F(\hat{t},\hat{y},\alpha(\hat{x}-\hat{y}) - 2\varepsilon\hat{y}, Y - 2\varepsilon I) &\ge 0. \end{aligned}$$

Using $a = b$, this implies

$$0 \le c \le F(\hat{t},\hat{x},\alpha(\hat{x}-\hat{y}) + 2\varepsilon\hat{x}, X + 2\varepsilon I) - F(\hat{t},\hat{y},\alpha(\hat{x}-\hat{y}) - 2\varepsilon\hat{y}, Y - 2\varepsilon I).$$

The last step consists in showing that the right-hand-side converges to zero by first sending $\varepsilon \to 0$ and then $\alpha \to \infty$. (This yields the desired contradiction which ends the proof.) If $\varepsilon$ were absent (e.g. set $\varepsilon = 0$ throughout) we would estimate

$$F(\hat{t},\hat{x},\alpha(\hat{x}-\hat{y}),X) - F(\hat{t},\hat{y},\alpha(\hat{x}-\hat{y}),Y) \le \theta\left(\alpha|\hat{x}-\hat{y}|^2 + |\hat{x}-\hat{y}|\right)$$

and conclude (using [6, lemma 3.1]) that

$$\alpha|\hat{x}-\hat{y}|^2, |\hat{x}-\hat{y}| \to 0 \text{ as } \alpha \to \infty$$

in conjunction with continuity of $\theta$ at $0+$. The present case, $\varepsilon > 0$, is essentially reduced to the case $\varepsilon = 0$ by adding/subtracting

$$F(\hat{t},\hat{x},\alpha(\hat{x}-\hat{y}),X) - F(\hat{t},\hat{y},\alpha(\hat{x}-\hat{y}),Y).$$

It follows that $c \le (i) + (ii) + (ii)$ where

$$\begin{aligned} (i) &= \left|F(\hat{t},\hat{x},\alpha(\hat{x}-\hat{y}) + 2\varepsilon\hat{x}, X + 2\varepsilon I) - F(\hat{t},\hat{x},\alpha(\hat{x}-\hat{y}),X)\right| \\ (ii) &= \left|F(\hat{t},\hat{y},\alpha(\hat{x}-\hat{y}) - 2\varepsilon\hat{y}, Y - 2\varepsilon I) - F(\hat{t},\hat{y},\alpha(\hat{x}-\hat{y}),Y)\right| \\ (iii) &= \theta\left(\alpha|\hat{x}-\hat{y}|^2 + |\hat{x}-\hat{y}|\right). \end{aligned}$$

From lemma 3, we see that (a) $p = \alpha(\hat{x}-\hat{y})$ remains, for fixed $\alpha$, bounded as $\varepsilon \to 0$, (b) $2\varepsilon|\hat{x}|$ and $2\varepsilon|\hat{y}|$ tend to zero as $\varepsilon \to 0$, for fixed $\alpha$, and (c)

$$\alpha|\hat{x}-\hat{y}|^2 + |\hat{x}-\hat{y}| \to 0 \quad \text{as } (1/\alpha,\varepsilon) \to (0,0).$$

We also note that (2.2) implies that any matrix norm of $X,Y$ is bounded by a constant times $\alpha$, independent of $\varepsilon$. Combining all this information shows that, fixed $\alpha$,

$$\lim_{\varepsilon\to0}(i) = \lim_{\varepsilon\to0}(ii) = 0$$

while $(iii) \to 0$ as $\varepsilon \to 0$ and $\alpha \to \infty$. In summary,

$$0 < c \leq \lim_{\alpha\to\infty} \limsup_{\varepsilon\to} [(i) + (ii) + (ii)] = 0$$

which is the desired contradiction. The proof is now finished. □

**Lemma 2.** *Assume* $u_0, v_0 \in \mathrm{BUC}(\mathbb{R}^n)$. *Then*

$$\sup_{x,y}\left[u_0(x) - v_0(y) - \frac{\alpha}{2}|x-y|^2 - \varepsilon\left(|x|^2 + |y|^2\right)\right] \to \sup_x [u_0(x) - v_0(x)] \quad \textit{as } \varepsilon \to 0, \alpha \to \infty.$$

*Proof.* Without loss of generality $M := \sup_x [u_0(x) - v_0(x)] > 0$; for otherwise replace $u_0$ by $u_0 + 2|M|$. Write $M_{\alpha,\varepsilon}$ for the achieved maximum (at $\hat{x}, \hat{y}$, say) of the left-hand-side. Obviously, $u_0(x) - v_0(x) - 2\varepsilon|x|^2 \leq M_{\alpha,\varepsilon}$ for any $x$ and so

$$M \leq \lim \inf_{\substack{\varepsilon\to 0\\ \alpha\to\infty}} M_{\alpha,\varepsilon}.$$

(It follows that we can and will consider $\varepsilon$ $(\alpha)$ small (large) enough so that $M_{\alpha,\varepsilon} > 0$.) On the other hand, $|u_0|, |v_0| \leq R < \infty$ and so

$$0 \leq M_{\alpha,\varepsilon} \leq 2R - \frac{\alpha}{2}|\hat{x} - \hat{y}|^2 - \varepsilon\left(|\hat{x}|^2 + |\hat{y}|^2\right)$$

from which we deduce $\frac{\alpha}{2}|\hat{x} - \hat{y}|^2 \leq 2R$, or $|\hat{x} - \hat{y}| \leq \sqrt{4R/\alpha}$. By omitting the (positive) penaltity terms, we can also estimate

$$\begin{aligned} M_{\alpha,\varepsilon} &\leq u_0(\hat{x}) - v_0(\hat{y}) \\ &= u_0(\hat{x}) - v_0(\hat{x}) + \sigma_{v_0}\left(\sqrt{4R/\alpha}\right) \\ &\leq M + \sigma_{v_0}\left(\sqrt{4R/\alpha}\right) \end{aligned}$$

where $\sigma_{v_0}$ denotes the modulus of continuity of $v_0$. It follows that

$$\lim \sup_{\substack{\varepsilon\to 0\\ \alpha\to\infty}} M_{\alpha,\varepsilon} \leq M$$

which shows that the $\lim M_{\alpha,\varepsilon}$ (as $\varepsilon \to 0, \alpha \to \infty$) exists and is equal to $M$. □

**Lemma 3.** *Consider a maximum point* $\left(\hat{t}, \hat{x}, \hat{y}\right) \in [0,T) \times \mathbb{R}^n \times \mathbb{R}^n$ *of*

$$\phi(t,x,y) = u(t,x) - v(t,y) - \frac{\alpha}{2}|x-y|^2 - \varepsilon\left(|x|^2 + |y|^2\right)$$

*where* $u, v \in \mathrm{BUC}([0,T] \times \mathbb{R}^n)$. *Then*

$$\begin{aligned} \limsup_{\varepsilon\to 0} \alpha(\hat{x} - \hat{y}) &= C_\alpha < \infty, \\ \lim_{\varepsilon\to 0} \varepsilon(|\hat{x}| + |\hat{y}|) &= 0 \\ \lim_{\substack{\varepsilon\to 0\\ \alpha\to\infty}} \left(\frac{\alpha}{2}|\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}|\right) &= 0. \end{aligned}$$

*Proof.* Without loss of generality $M := \sup_{[0,T]\times R^n} [u(t,x) - v(t,x)] > 0$; for otherwise replace $u$ by $u + 2|M|$. Write $M_{\alpha,\varepsilon}$ for the (achieved) maximum at some $\hat{t}, \hat{x}, \hat{y}$. Obviously, $u(t,x) - v(t,x) - 2\varepsilon|x|^2 \leq M_{\alpha,\varepsilon}$ for any $(t,x) \in [0,T] \times R^n$ and so

$$M \leq \lim \inf_{\substack{\varepsilon\to 0\\ \alpha\to\infty}} M_{\alpha,\varepsilon}.$$

(It follows that we can and will consider $\varepsilon$ ($\alpha$) small (large) enough so that $M_{\alpha,\varepsilon} > 0$.) On the other hand, $\sup_{[0,T]\times\mathbb{R}^n} |u(t,x)|, \sup_{[0,T]\times\mathbb{R}^n} |v(t,x)| \leq R < \infty$ and so

$$0 \leq M_{\alpha,\varepsilon} \leq 2R - \frac{\alpha}{2}|\hat{x}-\hat{y}|^2 - \varepsilon\left(|\hat{x}|^2 + |\hat{y}|^2\right).$$

(This allows already to deduce

$$\frac{\alpha}{2}|\hat{x}-\hat{y}|^2,\ \ \varepsilon\left(|\hat{x}|^2 + |\hat{y}|^2\right) \leq 2R$$

which readily implies $\alpha|\hat{x}-\hat{y}| \leq \sqrt{4\alpha R}$ and $\varepsilon|\hat{x}|, \varepsilon|\hat{y}| \leq \sqrt{2R\varepsilon}$.) Let us proceed as in the proof of lemma 2 and estimate

$$\begin{aligned} M_{\alpha,\varepsilon} &\leq u\left(\hat{t},\hat{x}\right) - v\left(\hat{t},\hat{y}\right) \\ &= u\left(\hat{t},\hat{x}\right) - v\left(\hat{t},\hat{x}\right) + \sigma_v\left(\sqrt{4R/\alpha}\right) \\ &\leq M + \sigma_v\left(\sqrt{4R/\alpha}\right) \end{aligned}$$

from which we see that $M_{\alpha,\varepsilon} \to M$ as $\varepsilon \to 0, \alpha \to \infty$. Necessarily then,

$$u\left(\hat{t},\hat{x}\right) - v\left(\hat{t},\hat{y}\right) - M_{\alpha,\varepsilon} = \frac{\alpha}{2}|\hat{x}-\hat{y}|^2 + \varepsilon\left(|\hat{x}|^2 + |\hat{y}|^2\right) \to 0$$

as $\varepsilon \to 0, \alpha \to \infty$. In particular,

$$\lim_{\substack{\varepsilon\to 0\\ \alpha\to\infty}} \frac{\alpha}{2}|\hat{x}-\hat{y}|^2 \to 0.$$

Since $|\hat{x}-\hat{y}| \to 0$ as $\varepsilon \to 0, \alpha \to \infty$ is clear from $|\hat{x}-\hat{y}| \leq \sqrt{4R/\alpha}$ the proof of the last estimate is also finished and the proof of the lemma is complete. □

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